

חדוא 1

פרק 35 - תרגילי תיאוריה מתקדמים - חשבון דיפרנציאלי (הפרק באנגלית)

תוכן העניינים

1. סדרות.....1
2. גבולות ורציפות.....3
3. משפט ערך הביניים ומשפט ויירשטראס.....4
4. גזירות ומשפט רול.....5
5. משפט לגראנז' וכלל לופיטל.....7
6. טורי חזקות וטורי טיילור.....10

Convergence of a Sequence, Monotone Sequences (סדרות)

Questions

- 1) Let A be a non-empty subset of $\mathbb{R}$ and $\alpha = \inf A$. Show that there exists a sequence (a_n) such that an $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \rightarrow \alpha$.
- 2) Let $x_0 \in \mathbb{Q}$. Show that there exists a sequence (x_n) of irrational numbers such that $x_n \rightarrow x_0$.
- 3) Let A be a non-empty subset of $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Show that there exists a sequence (a_n) in A such that $|x_0 - a_n| \rightarrow d(x_0, A)$. Recall that $d(x, A) = \inf \{|x - a| : a \in A\}$.
- 4) Let (a_k) be a bounded sequence. For every $n \in \mathbb{N}$, define $x_n = \sup\{a_k : k < n\}$. Show that the sequence (x_n) converges.

Cauchy Criterion, Bolzano - Weierstrass Theorem

- 5) Let (x_n) be a sequence of integers such that $|x_{n+1} - x_n| \geq 1$ for all $n \in \mathbb{N}$.
Prove or disprove the following statements.
 - a) The sequence (x_n) does not satisfy the Cauchy criterion.
 - b) The sequence (x_n) cannot have a convergent subsequence.
- 6) Show that a sequence (x_n) of real numbers has no convergent subsequence if and only if $|x_n| \rightarrow \infty$.
- 7) Let (x_n) be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose that every subsequence of (x_n) has a subsequence converging to x_0 . Show that $x_n \rightarrow x_0$.

- 8) Let (x_n) be a sequence in $\mathbb{R}$. We say that a positive integer n is a peak of the sequence if $m > n$ implies $x_n > x_m$ (i.e., if x_n is greater than every subsequent term in the sequence).
- a) If (x_n) has infinitely many peaks, show that it has a decreasing subsequence.
 - b) If (x_n) has only finitely many peaks, show that it has an increasing subsequence.
 - c) From (a) and (b) conclude that every sequence in $\mathbb{R}$ has a monotone subsequence. Further, every bounded sequence in $\mathbb{R}$ has a convergent subsequence (An alternate proof of Bolzano-Weierstrass Theorem).

Continuity and Limits (גבולות ורציפות)

Questions

- 1) Let $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 5$. Show that $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0$.
- 2) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose $\lim_{x \rightarrow x_0} f(x)$ exists.
Show that $\lim_{x \rightarrow 0} f(x + x_0) = \lim_{x \rightarrow x_0} f(x)$.
- 3) Let $f(x) = |x|$ for every $x \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.
- 4) Let $f: [0, \pi] \rightarrow \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = x \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$ for $x \neq 0$.
Is f continuous?
- 5) Let $[\cdot]$ denote the integer part function and $f: [0, \infty) \rightarrow \mathbb{R}$ be defined by $f(x) = [x^2] \sin \pi x$.
 - a) Show that f is continuous at each $x \neq \sqrt{n}$, $n \in \mathbb{N}$. [Here $\mathbb{N}$ includes 0]
 - b) Show that f is continuous at each $x = k \in \mathbb{N}$.
 - c) Show that f is discontinuous at each $x = \sqrt{n}$, $n \in \mathbb{N}$ such that $x \notin \mathbb{N}$.
- 6) Let the function $f: [0, 1] \rightarrow [a, b]$ be one-one and onto. Suppose f is continuous.
Show that f^{-1} is also continuous.
- 7) Let $f: (0, 1) \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q \in \mathbb{N} \text{ and } p, q \text{ have no common factor} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$
 - a) Suppose $x_n \rightarrow x_0$ for some x_0 , with $x_n \neq x$ for all $n \in \mathbb{N}$, and suppose $x_n = \frac{p_n}{q_n} \in (0, 1)$ where $p_n, q_n \in \mathbb{N}$ have no common factors. Show that $\lim_{n \rightarrow \infty} q_n = \infty$.
 - b) Show that f is continuous at every irrational.
 - c) Show that f is discontinuous at every rational.

Existence of Extrema, Intermediate Value Property (משפט ערך הביניים ומשפט ויירשטראס)

Questions

- 1) Give an example of a function f on $[0,1]$ which is not continuous but satisfies the IVP*. *We say that f has the property IVP [Intermediate Value Property] on $[a,b]$ if for every $x, y \in [a,b]$ and α satisfying $f(x) < \alpha < f(y)$ or $f(x) > \alpha > f(y)$ there exists $x_0 \in [x, y]$, such that $f(x_0) = \alpha$.
- 2) Let $f : [0,1] \rightarrow \mathbb{R}$ be continuous. Show that there exists an $x_0 \in [0,1]$ such that $f(x_0) = \frac{1}{3} \left[f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) \right]$.
- 3) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Show that f is a constant function if
 - a) $f(x)$ is rational for each $x \in \mathbb{R}$.
 - b) $f(x)$ is an integer for each $x \in \mathbb{Q}$.
- 4) Let $p(x) : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial function of odd degree. Show that p is onto.
- 5) Let $f, g : [0,1] \rightarrow \mathbb{R}$ be continuous such that $\inf\{f(x) : x \in [0,1]\} = \inf\{g(x) : x \in [0,1]\}$.
Show that there exists $x_0 \in [0,1]$ such that $f(x_0) = g(x_0)$.
- 6) A cross country runner runs continuously an eight kilometers course in 40 minutes without taking rest. Show that, somewhere along the course, the runner must have covered a distance of one kilometer in exactly 5 minutes.
- 7) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous one-one map.
Show that f is either strictly increasing or strictly decreasing.
- 8) Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a bijective map. Show that f is not continuous on $\mathbb{R}$.
- 9) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function.
 - a) Suppose f attains each of values exactly two times. Given:
 $f(x_1) = f(x_2) = \alpha$ for some $x_1, x_2, \alpha \in \mathbb{R}$, and $f(x_0) > \alpha$ for some $x_0 \in [x_1, x_2]$. Show that f attains its maximum in $[x_1, x_2]$ exactly at one point.
 - b) Using (a) show that f cannot attain each of its values exactly two times.

Differentiability and Rolle's Theorem

(גזירות ומשפט רול)

Questions

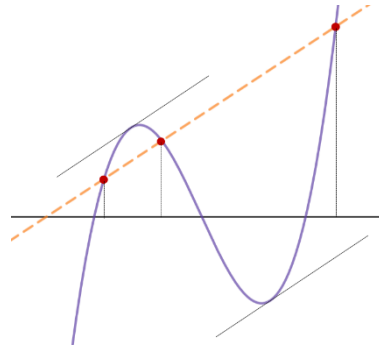
- 1) Let $f : (-1, 2) \rightarrow \mathbb{R}$ be a twice differentiable function such that $f''(0) > 0$.
Show that there exists $n \in \mathbb{N}$ that $f(\frac{1}{n}) \neq 1$.
- 2) Let f be a differentiable function on $\mathbb{R}$ such that $f(0) = f(1) = 0$, $f'(0) > 0$ and $f'(1) > 0$.
 - a) Show that $\exists \delta > 0$ such that $f(x) < 0$ on $(1-\delta, 1)$.
 - b) Show that $\exists c_1, c_2 \in (0, 1)$ such that $c_1 \neq c_2$ and $f'(c_1) = f'(c_2) = 0$.
- 3) Let $f : (0, 1) \rightarrow \mathbb{R}$ be thrice differentiable. Suppose $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. Show that there exists $x_0 \in (0, 1)$ such that $f'''(x_0) = 0$.
- 4) Give an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is differentiable only at $x = 1$.
- 5) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$.
 - a) If $f(x_0) \neq 0$, show that $|f|$ is also differentiable at x_0 .
 - b) If $f(x_0) = 0$, give examples to show that $|f|$ may or may not be differentiable at x_0 .
- 6) Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$. Define $h(x) = \max\{f(x), g(x)\} \forall x \in \mathbb{R}$. Show that if $f(x_0) \neq g(x_0)$ then h is differentiable at x_0 .
- 7) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable at $x = 1$ and $f(1) = 1$. Show that if $k \in \mathbb{N}$ then
$$\lim_{n \rightarrow \infty} n \left[f\left(1 + \frac{1}{n}\right) + f\left(1 + \frac{2}{n}\right) + \dots + f\left(1 + \frac{k}{n}\right) - k \right] = \frac{k(k+1)}{2} f'(1).$$
- 8) Let $f : [0, 1] \rightarrow \mathbb{R}$ be differentiable and $f(0) = 0$ and $f(1) = 1$.
Show that the equation $f'(x) = 2x$ has a solution on $(0, 1)$.

- 9) Let $f : [a, b] \rightarrow \mathbb{R}$ be such that $f'''(x)$ exists for all $x \in [a, b]$.
 Suppose that $f(a) = f(b) = f'(a) = f'(b) = 0$.
 Show that the equation $f'''(x) = 0$ has a solution.
- 10) Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable functions. Suppose that
 $\forall x \in \mathbb{R}, f'(x)g(x) \neq f(x)g'(x)$. Show that between any two roots of f , there
 exists at least one root* of g .
 *Recall: a root of f is a solution of $f(x) = 0$.
- 11) Let $f : (0, \infty) \rightarrow \mathbb{R}$ satisfy $f(xy) = f(x) + f(y) \quad \forall x, y \in (0, \infty)$.
 Suppose that f is differentiable at $x = 1$.
 Show that f is differentiable at every $x \in (0, \infty)$ and $f'(x) = \frac{f'(1)}{x}$.
 Hint: first show that $f(1) = 0$ and $f(\frac{x}{y}) = f(x) - f(y)$.
- 12) Let $p(x) = a + bx + cx^2$. Find all values of $a, b, c \in \mathbb{R}$ for which the function
 $p(|x|)$ is differentiable at 0.

Mean Value Theorem, L'Hôpital's Rule (משפט לגראנז' וכלל לופיטל)

Questions

- Does there exist a differentiable function $f:[0,2] \rightarrow \mathbb{R}$ satisfying $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$, for all $x \in [0,2]$?
- Let $f:[0,1] \rightarrow \mathbb{R}$ be differentiable such that $|f'(x)| < 1$ for all $x \in [0,1]$. Show that there exists at most one $c \in [0,1]$ such that $f(c) = c$.
- Let $f:\mathbb{R} \rightarrow \mathbb{R}$ be differentiable such that for some $\alpha \in \mathbb{R}$, $|f'(x)| \leq \alpha < 1$ for all $x \in \mathbb{R}$. Let $a_1 \in \mathbb{R}$ and define a sequence (a_n) recursively by $a_{n+1} = f(a_n)$. Show that (a_n) converges.
- Let $f:[0,1] \rightarrow \mathbb{R}$ be twice differentiable. Suppose that the line segment joining the points $(0, f(0))$ and $(1, f(1))$ intersect the graph of f at a point $(a, f(a))$, where $0 < a < 1$. Show that there exists $x_0 \in [0,1]$ such that $f''(x_0) = 0$.
- Let $f:[0,1] \rightarrow \mathbb{R}$ be continuous. Suppose f is differentiable on $(0,1)$ and $\lim_{x \rightarrow 0^+} f'(x) = L$ for some $L \in \mathbb{R}$. Show that $f'(0)$ exists and $f'(0) = L$.
- Let $f:[0,1] \rightarrow \mathbb{R}$ be differentiable and $f(0) = 0$. Suppose that $|f'(x)| < |f(x)|$ for all $x \in [0,1]$. Show that $f(x) = 0$ for all $x \in [0,1]$.
- Let $f:[0,\infty) \rightarrow \mathbb{R}$ be continuous and $f(0) = 0$. Suppose that $f'(x)$ exists for all $x \in (0,\infty)$ and f' is increasing on $(0,\infty)$. Show that the function $g(x) = \frac{f(x)}{x}$ is increasing on $(0,\infty)$.



8) Let $a \geq 0$ and $f : [a, b] \rightarrow \mathbb{R}$ be differentiable. Using Cauchy's mean value

theorem*, show that there exist $c_1, c_2 \in (a, b)$ such that $\frac{f'(c_1)}{a+b} = \frac{f'(c_2)}{2c_2}$.

$$* \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \quad a < c < b$$

9) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f''(c)$ exists at some $c \in \mathbb{R}$. Using L'Hôpital's

rule, show that $\lim_{h \rightarrow 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$. Give an example

where the above limit exists but $f''(c)$ does not exist.

10) Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable. If $f'(x) \neq 0$ for all $x \in [a, b]$, show that either $f'(x) \geq 0 \quad \forall x \in [a, b]$ or $f'(x) \leq 0 \quad \forall x \in [a, b]$.

11) Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable and let $\alpha \in \mathbb{R}$ be such that

$f'(a) < \alpha < f'(b)$. Define $g(x) = f(x) - \alpha x$ for all $x \in [a, b]$.

a) Show that there exists $c \in [a, b]$ such that $g'(c) = 0$.

Hint: prove by contradiction, noting that $g'(a) < 0$ and $g'(b) < 0$.

b) From the above, conclude that if a function f is differentiable on an interval $[a, b]$, then f' has the Intermediate Value Property on $[a, b]$.

12) Let f be differentiable on $[a, b]$ ($a < b$). Show that there exist

$c_1, c_2, c_3 \in (a, b)$ such that $c_2 \neq c_3$ and $f'(c_2) + f'(c_3) = 2f'(c_1)$.

13) Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is continuous and $\int_0^1 f(t) dt = 1$.

a) Show that there exists $c \in (0, 1)$ such that $f(c) = 1$.

b) Show that there exist $c_1 \neq c_2$ in $(0, 1)$ such that $f(c_1) + f(c_2) = 2$.

14) Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $|f'(x)| < 10$ for all $x \in (0, 1)$ and let (x_n) be a sequence in $(0, 1)$ satisfying the Cauchy criterion. Show that the sequence $(f(x_n))$ converges.

15) Let $f : [0,1] \rightarrow [0,1]$ be such that $f'(x) < 0$ for all $x \in [0,1]$. Show that there is one and only one $c \in [0,1]$ such that $f(c) = c^2$.

16) Let $f : [0,1] \rightarrow \mathbb{R}$ and $a_n = f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)$, $n=1,2,\dots$

Show that:

a) if f is continuous, then $\sum_{n=1}^{\infty} a_n$ converges;

b) if f is differentiable and $|f'(x)| < \frac{1}{2} \forall x \in [0,1]$, then

$\sum_{n=1}^{\infty} a_n (\cos n) \sqrt{n}$ converges.

Power Series, Taylor Series (טורי חזקות וטורי טיילור)

Questions

1) Answer the following sections:

a) Let $f:[a,b] \rightarrow \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a,b]$.

Suppose $x_0 \in [a,b]$.

Show that for any $x \in [a,b]$ $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$.

I.e., the graph of f lies above the tangent line to the graph at $(x_0, f(x_0))$.

b) Show that $\cos y - \cos x \geq (x - y)\sin x$ for all $x, y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

2) Let $f:(a,b) \rightarrow \mathbb{R}$ be infinitely differentiable and let $x_0 \in (a,b)$. Suppose that there exists $M > 0$ such that $|f^{(n)}(x)| \leq M^n$ for all $n \in \mathbb{N}$ and $x \in (a,b)$.

Show that Taylor's series of f around x_0 converges to $f(x)$ for all $x \in (a,b)$.

3) Let (a_n) be a sequence of nonnegative reals and suppose that $(a_n^{\frac{1}{n}})$ is a

bounded sequence. For each n , define $A_n = \sup\{a_k^{\frac{1}{k}} : k \geq n\}$.

(A_n) converges since it is decreasing and bounded below (by 0).

So $A_n \rightarrow L$ for some $L \geq 0$.

a) Show that if $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges and if $L > 1$ the series diverges.

b) Show that the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$ is $\frac{1}{L}$.