

# חדוא 1

פרק 38 - תרגילי תיאוריה מתקדמים - חשבון דיפרנציאלי (הפרק באנגלית)

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## Convergence of a Sequence, Monotone Sequences (סדרות)

### Questions

- 1) Let  $A$  be a non-empty subset of  $\mathbb{R}$  and  $\alpha = \inf A$ . Show that there exists a sequence  $(a_n)$  such that an  $a_n \in A$  for all  $n \in \mathbb{N}$  and  $a_n \rightarrow \alpha$ .
- 2) Let  $x_0 \in \mathbb{Q}$ . Show that there exists a sequence  $(x_n)$  of irrational numbers such that  $x_n \rightarrow x_0$ .
- 3) Let  $A$  be a non-empty subset of  $\mathbb{R}$  and  $x_0 \in \mathbb{R}$ . Show that there exists a sequence  $(a_n)$  in  $A$  such that  $|x_0 - a_n| \rightarrow d(x_0, A)$ . Recall that  $d(x, A) = \inf \{|x - a| : a \in A\}$ .
- 4) Let  $(a_k)$  be a bounded sequence. For every  $n \in \mathbb{N}$ , define  $x_n = \sup\{a_k : k < n\}$ . Show that the sequence  $(x_n)$  converges.

### Cauchy Criterion, Bolzano - Weierstrass Theorem

- 5) Let  $(x_n)$  be a sequence of integers such that  $|x_{n+1} - x_n| \geq 1$  for all  $n \in \mathbb{N}$ .  
 Prove or disprove the following statements.
  - a) The sequence  $(x_n)$  does not satisfy the Cauchy criterion.
  - b) The sequence  $(x_n)$  cannot have a convergent subsequence.
- 6) Show that a sequence  $(x_n)$  of real numbers has no convergent subsequence if and only if  $|x_n| \rightarrow \infty$ .
- 7) Let  $(x_n)$  be a sequence in  $\mathbb{R}$  and  $x_0 \in \mathbb{R}$ . Suppose that every subsequence of  $(x_n)$  has a subsequence converging to  $x_0$ . Show that  $x_n \rightarrow x_0$ .

- 8) Let  $(x_n)$  be a sequence in  $\mathbb{R}$ . We say that a positive integer  $n$  is a peak of the sequence if  $m > n$  implies  $x_n > x_m$  (i.e., if  $x_n$  is greater than every subsequent term in the sequence).
- If  $(x_n)$  has infinitely many peaks, show that it has a decreasing subsequence.
  - If  $(x_n)$  has only finitely many peaks, show that it has an increasing subsequence.
  - From (a) and (b) conclude that every sequence in  $\mathbb{R}$  has a monotone subsequence. Further, every bounded sequence in  $\mathbb{R}$  has a convergent subsequence (An alternate proof of Bolzano-Weierstrass Theorem).

## Continuity and Limits (גבולות ורציפות)

### Questions

- 1) Let  $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 5$ . Show that  $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0$ .
- 2) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $x_0 \in \mathbb{R}$ . Suppose  $\lim_{x \rightarrow x_0} f(x)$  exists.  
 Show that  $\lim_{x \rightarrow 0} f(x + x_0) = \lim_{x \rightarrow x_0} f(x)$ .
- 3) Let  $f(x) = |x|$  for every  $x \in \mathbb{R}$ . Show that  $f$  is continuous on  $\mathbb{R}$ .
- 4) Let  $f : [0, \pi] \rightarrow \mathbb{R}$  be defined by  $f(0) = 0$  and  $f(x) = x \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$  for  $x \neq 0$ .  
 Is  $f$  continuous?
- 5) Let  $[\cdot]$  denote the integer part function and  $f : [0, \infty) \rightarrow \mathbb{R}$  be defined by  
 $f(x) = [x^2] \sin \pi x$ .
  - a) Show that  $f$  is continuous at each  $x \neq \sqrt{n}$ ,  $n \in \mathbb{N}$ . [Here  $\mathbb{N}$  includes 0]
  - b) Show that  $f$  is continuous at each  $x = k \in \mathbb{N}$ .
  - c) Show that  $f$  is discontinuous at each  $x = \sqrt{n}$ ,  $n \in \mathbb{N}$  such that  $x \notin \mathbb{N}$ .
- 6) Let the function  $f : [0, 1] \rightarrow [a, b]$  be one-one and onto. Suppose  $f$  is continuous.  
 Show that  $f^{-1}$  is also continuous.
- 7) Let  $f : (0, 1) \rightarrow \mathbb{R}$  be given by
 
$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q \in \mathbb{N} \text{ and } p, q \text{ have no common factor} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$
  - a) Suppose  $x_n \rightarrow x_0$  for some  $x_0$ , with  $x_n \neq x_0$  for all  $n \in \mathbb{N}$ , and suppose  
 $x_n = \frac{p_n}{q_n} \in (0, 1)$  where  $p_n, q_n \in \mathbb{N}$  have no common factors. Show that  
 $\lim_{n \rightarrow \infty} q_n = \infty$ .
  - b) Show that  $f$  is continuous at every irrational.
  - c) Show that  $f$  is discontinuous at every rational.

## Existence of Extrema, Intermediate Value Property (משפט ערך הביניים ומשפט ויירשטראס)

### Questions

- 1) Give an example of a function  $f$  on  $[0,1]$  which is not continuous but satisfies the IVP\*. \*We say that  $f$  has the property IVP [Intermediate Value Property] on  $[a,b]$  if for every  $x, y \in [a,b]$  and  $\alpha$  satisfying  $f(x) < \alpha < f(y)$  or  $f(x) > \alpha > f(y)$  there exists  $x_0 \in [x, y]$ , such that  $f(x_0) = \alpha$ .
- 2) Let  $f : [0,1] \rightarrow \mathbb{R}$  be continuous. Show that there exists an  $x_0 \in [0,1]$  such that  $f(x_0) = \frac{1}{3} [f(\frac{1}{4}) + f(\frac{1}{2}) + f(\frac{3}{4})]$ .
- 3) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function. Show that  $f$  is a constant function if
  - a)  $f(x)$  is rational for each  $x \in \mathbb{R}$ .
  - b)  $f(x)$  is an integer for each  $x \in \mathbb{Q}$ .
- 4) Let  $p(x) : \mathbb{R} \rightarrow \mathbb{R}$  be a polynomial function of odd degree. Show that  $p$  is onto.
- 5) Let  $f, g : [0,1] \rightarrow \mathbb{R}$  be continuous such that  $\inf\{f(x) : x \in [0,1]\} = \inf\{g(x) : x \in [0,1]\}$ . Show that there exists  $x_0 \in [0,1]$  such that  $f(x_0) = g(x_0)$ .
- 6) A cross country runner runs continuously an eight kilometers course in 40 minutes without taking rest. Show that, somewhere along the course, the runner must have covered a distance of one kilometer in exactly 5 minutes.
- 7) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous one-one map. Show that  $f$  is either strictly increasing or strictly decreasing.
- 8) Let  $f : \mathbb{R} \rightarrow [0, \infty)$  be a bijective map. Show that  $f$  is not continuous on  $\mathbb{R}$ .
- 9) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function.
  - a) Suppose  $f$  attains each of values exactly two times. Given:  $f(x_1) = f(x_2) = \alpha$  for some  $x_1, x_2, \alpha \in \mathbb{R}$ , and  $f(x_0) > \alpha$  for some  $x_0 \in [x_1, x_2]$ . Show that  $f$  attains its maximum in  $[x_1, x_2]$  exactly at one point.
  - b) Using (a) show that  $f$  cannot attain each of its values exactly two times.

## Differentiability and Rolle's Theorem

### (גזירות ומשפט רול)

### Questions

- 1) Let  $f : (-1, 2) \rightarrow \mathbb{R}$  be a twice differentiable function such that  $f''(0) > 0$ . Show that there exists  $n \in \mathbb{N}$  that  $f(\frac{1}{n}) \neq 1$ .
- 2) Let  $f$  be a differentiable function on  $\mathbb{R}$  such that  $f(0) = f(1) = 0$ ,  $f'(0) > 0$  and  $f'(1) > 0$ .
  - a) Show that  $\exists \delta > 0$  such that  $f(x) < 0$  on  $(1 - \delta, 1)$ .
  - b) Show that  $\exists c_1, c_2 \in (0, 1)$  such that  $c_1 \neq c_2$  and  $f'(c_1) = f'(c_2) = 0$ .
- 3) Let  $f : (0, 1) \rightarrow \mathbb{R}$  be thrice differentiable. Suppose  $f(\frac{1}{n}) = 0$  for all  $n \in \mathbb{N}$ . Show that there exists  $x_0 \in (0, 1)$  such that  $f'''(x_0) = 0$ .
- 4) Give an example of a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  which is differentiable only at  $x = 1$ .
- 5) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable at  $x_0 \in \mathbb{R}$ .
  - a) If  $f(x_0) \neq 0$ , show that  $|f|$  is also differentiable at  $x_0$ .
  - b) If  $f(x_0) = 0$ , give examples to show that  $|f|$  may or may not be differentiable at  $x_0$ .
- 6) Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable at  $x_0 \in \mathbb{R}$ . Define  $h(x) = \max\{f(x), g(x)\} \forall x \in \mathbb{R}$ . Show that if  $f(x_0) \neq g(x_0)$  then  $h$  is differentiable at  $x_0$ .
- 7) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable at  $x = 1$  and  $f(1) = 1$ . Show that if  $k \in \mathbb{N}$  then 
$$\lim_{n \rightarrow \infty} n \left[ f\left(1 + \frac{1}{n}\right) + f\left(1 + \frac{2}{n}\right) + \dots + f\left(1 + \frac{k}{n}\right) - k \right] = \frac{k(k+1)}{2} f'(1).$$
- 8) Let  $f : [0, 1] \rightarrow \mathbb{R}$  be differentiable and  $f(0) = 0$  and  $f(1) = 1$ . Show that the equation  $f'(x) = 2x$  has a solution on  $(0, 1)$ .

- 9) Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f'''(x)$  exists for all  $x \in [a, b]$ .  
 Suppose that  $f(a) = f(b) = f'(a) = f'(b) = 0$ .  
 Show that the equation  $f'''(x) = 0$  has a solution.
- 10) Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable functions. Suppose that  
 $\forall x \in \mathbb{R}, f'(x)g(x) \neq f(x)g'(x)$ . Show that between any two roots of  $f$ , there  
 exists at least one root\* of  $g$ .  
 \*Recall: a root of  $f$  is a solution of  $f(x) = 0$ .
- 11) Let  $f : (0, \infty) \rightarrow \mathbb{R}$  satisfy  $f(xy) = f(x) + f(y) \forall x, y \in (0, \infty)$ .  
 Suppose that  $f$  is differentiable at  $x = 1$ .  
 Show that  $f$  is differentiable at every  $x \in (0, \infty)$  and  $f'(x) = \frac{f'(1)}{x}$ .  
 Hint: first show that  $f(1) = 0$  and  $f(\frac{x}{y}) = f(x) - f(y)$ .
- 12) Let  $p(x) = a + bx + cx^2$ . Find all values of  $a, b, c \in \mathbb{R}$  for which the function  
 $p(|x|)$  is differentiable at 0.

## Mean Value Theorem, L'Hôpital's Rule

### (משפט לגראנז' וכלל לופיטל)

### Questions

- Does there exist a differentiable function  $f : [0, 2] \rightarrow \mathbb{R}$  satisfying  $f(0) = -1$ ,  $f(2) = 4$  and  $f'(x) \leq 2$ , for all  $x \in [0, 2]$ ?
- Let  $f : [0, 1] \rightarrow \mathbb{R}$  be differentiable such that  $|f'(x)| < 1$  for all  $x \in [0, 1]$ . Show that there exists at most one  $c \in [0, 1]$  such that  $f(c) = c$ .
- Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be differentiable such that for some  $\alpha \in \mathbb{R}$ ,  $|f'(x)| \leq \alpha < 1$  for all  $x \in \mathbb{R}$ . Let  $a_1 \in \mathbb{R}$  and define a sequence  $(a_n)$  recursively by  $a_{n+1} = f(a_n)$ . Show that  $(a_n)$  converges.
- Let  $f : [0, 1] \rightarrow \mathbb{R}$  be twice differentiable. Suppose that the line segment joining the points  $(0, f(0))$  and  $(1, f(1))$  intersect the graph of  $f$  at a point  $(a, f(a))$ , where  $0 < a < 1$ . Show that there exists  $x_0 \in [0, 1]$  such that  $f''(x_0) = 0$ .
 
- Let  $f : [0, 1] \rightarrow \mathbb{R}$  be continuous. Suppose  $f$  is differentiable on  $(0, 1)$  and  $\lim_{x \rightarrow 0^+} f'(x) = L$  for some  $L \in \mathbb{R}$ . Show that  $f'(0)$  exists and  $f'(0) = L$ .
- Let  $f : [0, 1] \rightarrow \mathbb{R}$  be differentiable and  $f(0) = 0$ . Suppose that  $|f'(x)| < |f(x)|$  for all  $x \in [0, 1]$ . Show that  $f(x) = 0$  for all  $x \in [0, 1]$ .
- Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be continuous and  $f(0) = 0$ . Suppose that  $f'(x)$  exists for all  $x \in (0, \infty)$  and  $f'$  is increasing on  $(0, \infty)$ . Show that the function  $g(x) = \frac{f(x)}{x}$  is increasing on  $(0, \infty)$ .

- 8) Let  $a \geq 0$  and  $f : [a, b] \rightarrow \mathbb{R}$  be differentiable. Using Cauchy's mean value theorem\*, show that there exist  $c_1, c_2 \in (a, b)$  such that  $\frac{f'(c_1)}{a+b} = \frac{f'(c_2)}{2c_2}$ .
- \*  $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, a < c < b$
- 9) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be such that  $f''(c)$  exists at some  $c \in \mathbb{R}$ . Using L'Hôpital's rule, show that  $\lim_{h \rightarrow 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$ . Give an example where the above limit exists but  $f''(c)$  does not exist.
- 10) Let  $f : [a, b] \rightarrow \mathbb{R}$  be differentiable. If  $f'(x) \neq 0$  for all  $x \in [a, b]$ , show that either  $f'(x) \geq 0 \forall x \in [a, b]$  or  $f'(x) \leq 0 \forall x \in [a, b]$ .
- 11) Let  $f : [a, b] \rightarrow \mathbb{R}$  be differentiable and let  $\alpha \in \mathbb{R}$  be such that  $f'(a) < \alpha < f'(b)$ . Define  $g(x) = f(x) - \alpha x$  for all  $x \in [a, b]$ .
- Show that there exists  $c \in [a, b]$  such that  $g'(c) = 0$ .  
Hint: prove by contradiction, noting that  $g'(a) < 0$  and  $g'(b) < 0$ .
  - From the above, conclude that if a function  $f$  is differentiable on an interval  $[a, b]$ , then  $f'$  has the Intermediate Value Property on  $[a, b]$ .
- 12) Let  $f$  be differentiable on  $[a, b]$  ( $a < b$ ). Show that there exist  $c_1, c_2, c_3 \in (a, b)$  such that  $c_2 \neq c_3$  and  $f'(c_2) + f'(c_3) = 2f'(c_1)$ .
- 13) Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  is continuous and  $\int_0^1 f(t) dt = 1$ .
- Show that there exists  $c \in (0, 1)$  such that  $f(c) = 1$ .
  - Show that there exist  $c_1 \neq c_2$  in  $(0, 1)$  such that  $f(c_1) + f(c_2) = 2$ .
- 14) Let  $f : [0, 1] \rightarrow \mathbb{R}$  be such that  $|f'(x)| < 10$  for all  $x \in (0, 1)$  and let  $(x_n)$  be a sequence in  $(0, 1)$  satisfying the Cauchy criterion. Show that the sequence  $(f(x_n))$  converges.

15) Let  $f : [0,1] \rightarrow [0,1]$  be such that  $f'(x) < 0$  for all  $x \in [0,1]$ . Show that there is one and only one  $c \in [0,1]$  such that  $f(c) = c^2$ .

16) Let  $f : [0,1] \rightarrow \mathbb{R}$  and  $a_n = f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)$ ,  $n = 1, 2, \dots$

Show that:

a) if  $f$  is continuous, then  $\sum_{n=1}^{\infty} a_n$  converges;

b) if  $f$  is differentiable and  $|f'(x)| < \frac{1}{2} \forall x \in [0,1]$ , then

$\sum_{n=1}^{\infty} a_n (\cos n) \sqrt{n}$  converges.

## Power Series, Taylor Series (טורי חזקות וטורי טיילור)

### Questions

1) Answer the following sections:

a) Let  $f : [a, b] \rightarrow \mathbb{R}$  be such that  $f''(x) \geq 0$  for all  $x \in [a, b]$ .

Suppose  $x_0 \in [a, b]$ .

Show that for any  $x \in [a, b]$   $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$ .

I.e., the graph of  $f$  lies above the tangent line to the graph at  $(x_0, f(x_0))$ .

b) Show that  $\cos y - \cos x \geq (x - y)\sin x$  for all  $x, y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ .

2) Let  $f : (a, b) \rightarrow \mathbb{R}$  be infinitely differentiable and let  $x_0 \in (a, b)$ . Suppose that there exists  $M > 0$  such that  $|f^{(n)}(x)| \leq M^n$  for all  $n \in \mathbb{N}$  and  $x \in (a, b)$ .

Show that Taylor's series of  $f$  around  $x_0$  converges to  $f(x)$  for all  $x \in (a, b)$ .

3) Let  $(a_n)$  be a sequence of nonnegative reals and suppose that  $(a_n^{\frac{1}{n}})$  is a bounded sequence. For each  $n$ , define  $A_n = \sup\{a_k^{\frac{1}{k}} : k \geq n\}$ .

$(A_n)$  converges since it is decreasing and bounded below (by 0).

So  $A_n \rightarrow L$  for some  $L \geq 0$ .

a) Show that if  $L < 1$ , the series  $\sum_{n=1}^{\infty} a_n$  converges and if  $L > 1$  the series diverges.

b) Show that the radius of convergence of the power series  $\sum_{n=1}^{\infty} a_n x^n$  is  $\frac{1}{L}$ .