חדו"א א

פרק 35 - תרגילי תיאוריה מתקדמים - חשבון דיפרנציאלי (הפרק באנגלית)

תוכן העניינים

Convergence of a Sequence, Monotone Sequences (סדרות)

Questions

1) Let A be a non-empty subset of $\mathbb{R}$ and $\alpha = \inf A$. Show that there exists a sequence (a_n) such that an $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \to \alpha$.

2) Let $x_0 \in \mathbb{Q}$. Show that there exists a sequence (x_n) of irrational numbers such that $x_n \to x_0$.

3) Let A be a non-empty subset of $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Show that there exists a sequence (a_n) in A such that $|x_0 - a_n| \to d(x_0, A)$. Recall that $d(x,A) = \inf\{|x-a| : a \in A\}$.

4) Let (a_k) be a bounded sequence. For every $n \in \mathbb{N}$, define $x_n = \sup\{a_k : k < n\}$. Show that the sequence (x_n) converges.

Cauchy Criterion, Bolzano - Weierstrass Theorem

5) Let (x_n) be a sequence of integers such that $|x_{n+1} - x_n| \geq 1$ for all $n \in \mathbb{N}$. Prove or disprove the following statements.

a) The sequence (x_n) does not satisfy the Cauchy criterion.

b) The sequence (x_n) cannot have a convergent subsequence.

6) Show that a sequence (x_n) of real numbers has no convergent subsequence if and only if $|x_n| \to \infty$.

7) Let (x_n) be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose that every subsequence of (x_n) has a subsequence converging to x_0. Show that $x_n \to x_0$.

8) Let (x_n) be a sequence in $\mathbb{R}$. We say that a positive integer *n* is a peak of the sequence if $m > n$ implies $x_n > x_m$ (i.e., if x_n is greater than every subsequent term in the sequence).

a) If (x_n) has infinitely many peaks, show that it has a decreasing subsequence.

b) If (x_n) has only finitely many peaks, show that it has an increasing subsequence.

c) From (a) and (b) conclude that every sequence in $\mathbb{R}$ has a monotone subsequence. Further, every bounded sequence in $\mathbb{R}$ has a convergent subsequence (An alternate proof of Bolzano-Weierstrass Theorem).

Continuity and Limits (גבולות ורציפות)

Questions

1) Let $\lim_{x\to 0}\frac{f(x)}{x^2}=5$. Show that $\lim_{x\to 0}\frac{f(x)}{x}=0$.

2) Let $f:\mathbb{R}\to\mathbb{R}$ and $x_0\in\mathbb{R}$. Suppose $\lim_{x\to x_0} f(x)$ exists.

Show that $\lim_{x\to 0} f(x+x_0)=\lim_{x\to x_0} f(x)$.

3) Let $f(x)=|x|$ for every $x\in\mathbb{R}$. Show that f is continuous on $\mathbb{R}$.

4) Let $f:[0,\pi]\to\mathbb{R}$ be defined by $f(0)=0$ and $f(x)=x\sin\frac{1}{x}-\frac{1}{x}\cos\frac{1}{x}$ for $x\neq 0$. Is f continuous?

5) Let $[\cdot]$ denote the integer part function and $f:[0,\infty)\to\mathbb{R}$ be defined by $f(x)=[x^2]\sin\pi x$.

a) Show that f is continuous at each $x\neq\sqrt{n}$, $n\in\mathbb{N}$. [Here $\mathbb{N}$ includes 0]

b) Show that f is continuous at each $x=k\in\mathbb{N}$.

c) Show that f is discontinuous at each $x=\sqrt{n}$, $n\in\mathbb{N}$ such that $x\notin\mathbb{N}$.

6) Let the function $f:[0,1]\to[a,b]$ be one-one and onto. Suppose f is continuous. Show that f^{-1} is also continuous.

7) Let $f:(0,1)\to\mathbb{R}$ be given by

$$f(x)=\begin{cases}\frac{1}{q} & \text{if } x=\frac{p}{q} \text{ where } p,q\in\mathbb{N} \text{ and } p,q \text{ have no common factor}\\ 0 & \text{if } x \text{ is irrational}\end{cases}$$

a) Suppose $x_n\to x_0$ for some x_0, with $x_n\neq x$ for all $n\in\mathbb{N}$, and suppose $x_n=\frac{p_n}{q_n}\in(0,1)$ where $p_n,q_n\in\mathbb{N}$ have no common factors. Show that $\lim_{n\to\infty} q_n=\infty$.

b) Show that f is continuous at every irrational.

c) Show that f is discontinuous at every rational.

Existence of Extrema, Intermediate Value Property (משפט ערך הביניים ומשפט ויירשטראס)

Questions

1) Give an example of a function f on $[0,1]$ which is not continuous but satisfies the IVP*. *We say that f has the property IVP [Intermediate Value Property] on $[a,b]$ if for every $x, y \in [a,b]$ and α satisfying $f(x) < \alpha < f(y)$ or $f(x) > \alpha > f(y)$ there exists $x_0 \in [x, y]$, such that $f(x_0) = \alpha$.

2) Let $f:[0,1] \to \mathbb{R}$ be continuous. Show that there exists an $x_0 \in [0,1]$ such that $f(x_0) = \frac{1}{3}\left[f(\frac{1}{4}) + f(\frac{1}{2}) + f(\frac{3}{4})\right]$.

3) Let $f:\mathbb{R} \to \mathbb{R}$ be a continuous function. Show that f is a constant function if
a) $f(x)$ is rational for each $x \in \mathbb{R}$.
b) $f(x)$ is an integer for each $x \in \mathbb{Q}$.

4) Let $p(x):\mathbb{R} \to \mathbb{R}$ be a polynomial function of odd degree. Show that p is onto.

5) Let $f, g:[0,1] \to \mathbb{R}$ be continuous such that
$\inf\{f(x): x \in [0,1]\} = \inf\{g(x): x \in [0,1]\}$.
Show that there exists $x_0 \in [0,1]$ such that $f(x_0) = g(x_0)$.

6) A cross country runner runs continuously an eight kilometers course in 40 minutes without taking rest. Show that, somewhere along the course, the runner must have covered a distance of one kilometer in exactly 5 minutes.

7) Let $f:\mathbb{R} \to \mathbb{R}$ be a continuous one-one map.
Show that f is either strictly increasing or strictly decreasing.

8) Let $f:\mathbb{R} \to [0,\infty)$ be a bijective map. Show that f is not continuous on $\mathbb{R}$.

9) Let $f:\mathbb{R} \to \mathbb{R}$ be a continuous function.
a) Suppose f attains each of values exactly two times. Given:
$f(x_1) = f(x_2) = \alpha$ for some $x_1, x_2, \alpha \in \mathbb{R}$, and $f(x_0) > \alpha$ for some $x_0 \in [x_1, x_2]$. Show that f attains its maximum in $[x_1, x_2]$ exactly at one point.
b) Using (a) show that f cannot attain each of its values exactly two times.

Differentiability and Rolle's Theorem
(גזירות ומשפט רול)

Questions

1) Let $f:(-1,2)\to\mathbb{R}$ be a twice differentiable function such that $f''(0)>0$. Show that there exists $n\in\mathbb{N}$ that $f(\frac{1}{n})\neq 1$.

2) Let f be a differentiable function on $\mathbb{R}$ such that $f(0)=f(1)=0$, $f'(0)>0$ and $f'(1)>0$.

a) Show that $\exists\delta>0$ such that $f(x)<0$ on $(1-\delta,1)$.

b) Show that $\exists c_1,c_2\in(0,1)$ such that $c_1\neq c_2$ and $f'(c_1)=f'(c_2)=0$.

3) Let $f:(0,1)\to\mathbb{R}$ be thrice differentiable. Suppose $f(\frac{1}{n})=0$ for all $n\in\mathbb{N}$. Show that there exists $x_0\in(0,1)$ such that $f'''(x_0)=0$.

4) Give an example of a function $f:\mathbb{R}\to\mathbb{R}$ which is differentiable only at $x=1$.

5) Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable at $x_0\in\mathbb{R}$.

a) If $f(x_0)\neq 0$, show that $|f|$ is also differentiable at x_0.

b) If $f(x_0)=0$, give examples to show that $|f|$ may or may not be differentiable at x_0.

6) Let $f,g:\mathbb{R}\to\mathbb{R}$ be differentiable at $x_0\in\mathbb{R}$. Define $h(x)=\max\{f(x),\ g(x)\}\ \forall x\in\mathbb{R}$. Show that if $f(x_0)\neq g(x_0)$ then h is differentiable at x_0.

7) Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable at $x=1$ and $f(1)=1$. Show that if $k\in\mathbb{N}$ then $\lim\limits_{n\to\infty} n\left[f(1+\frac{1}{n})+f(1+\frac{2}{n})+\ldots+f(1+\frac{k}{n})-k\right]=\frac{k(k+1)}{2}f'(1)$.

8) Let $f:[0,1]\to\mathbb{R}$ be differentiable and $f(0)=0$ and $f(1)=1$. Show that the equation $f'(x)=2x$ has a solution on $(0,1)$.

9) Let $f:[a,b]\to\mathbb{R}$ be such that $f'''(x)$ exists for all $x\in[a,b]$.
Suppose that $f(a)=f(b)=f'(a)=f'(b)=0$.
Show that the equation $f'''(x)=0$ has a solution.

10) Let $f,g:\mathbb{R}\to\mathbb{R}$ be differentiable functions. Suppose that $\forall x\in\mathbb{R},\ f'(x)g(x)\neq f(x)g'(x)$. Show that between any two roots of f, there exists at least one root* of g.
*Recall: a root of f is a solution of $f(x)=0$.

11) Let $f:(0,\infty)\to\mathbb{R}$ satisfy $f(xy)=f(x)+f(y)\ \forall x,y\in(0,\infty)$.
Suppose that f is differentiable at $x=1$.
Show that f is differentiable at every $x\in(0,\infty)$ and $f'(x)=\dfrac{f'(1)}{x}$.
Hint: first show that $f(1)=0$ and $f(\frac{x}{y})=f(x)-f(y)$.

12) Let $p(x)=a+bx+cx^2$. Find all values of $a,b,c\in\mathbb{R}$ for which the function $p(|x|)$ is differentiable at 0.

Mean Value Theorem, L'Hôpital's Rule (משפט לגראנז' וכלל לופיטל)

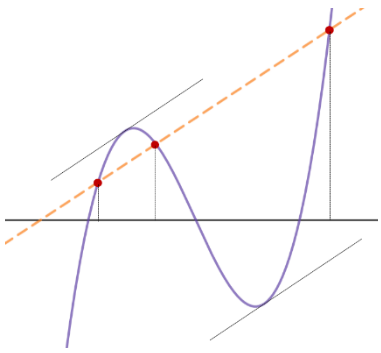
Questions

1) Does there exist a differentiable function $f:[0,2]\to\mathbb{R}$ satisfying $f(0)=-1,\ f(2)=4$ and $f'(x)\le 2$, for all $x\in[0,2]$?

2) Let $f:[0,1]\to\mathbb{R}$ be differentiable such that $|f'(x)|<1$ for all $x\in[0,1]$. Show that there exists at most one $c\in[0,1]$ such that $f(c)=c$.

3) Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable such that for some $\alpha\in\mathbb{R}$, $|f'(x)|\le\alpha<1$ for all $x\in\mathbb{R}$. Let $a_1\in\mathbb{R}$ and define a sequence (a_n) recursively by $a_{n+1}=f(a_n)$. Show that (a_n) converges.

4) Let $f:[0,1]\to\mathbb{R}$ be twice differentiable. Suppose that the line segment joining the points $(0,f(0))$ and $(1,f(1))$ intersect the graph of f at a point $(a,f(a))$, where $0<a<1$. Show that there exists $x_0\in[0,1]$ such that $f''(x_0)=0$.



5) Let $f:[0,1]\to\mathbb{R}$ be continuous. Suppose f is differentiable on $(0,1)$ and $\lim\limits_{x\to 0^+} f'(x)=L$ for some $L\in\mathbb{R}$. Show that $f'(0)$ exists and $f'(0)=L$.

6) Let $f:[0,1]\to\mathbb{R}$ be differentiable and $f(0)=0$. Suppose that $|f'(x)|<|f(x)|$ for all $x\in[0,1]$. Show that $f(x)=0$ for all $x\in[0,1]$.

7) Let $f:[0,\infty)\to\mathbb{R}$ be continuous and $f(0)=0$. Suppose that $f'(x)$ exists for all $x\in(0,\infty)$ and f' is increasing on $(0,\infty)$. Show that the function $g(x)=\frac{f(x)}{x}$ is increasing on $(0,\infty)$.

8) Let $a \geq 0$ and $f:[a,b] \to \mathbb{R}$ be differentiable. Using Cauchy's mean value theorem*, show that there exist $c_1, c_2 \in (a,b)$ such that $\frac{f'(c_1)}{a+b} = \frac{f'(c_2)}{2c_2}$.

* $\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \quad a<c<b}$

9) Let $f:\mathbb{R} \to \mathbb{R}$ be such that $f''(c)$ exists at some $c \in \mathbb{R}$. Using L'Hôpital's rule, show that $\lim\limits_{h\to 0} \frac{f(c+h)-2f(c)+f(c-h)}{h^2} = f''(c)$. Give an example where the above limit exists but $f''(c)$ does not exist.

10) Let $f:[a,b] \to \mathbb{R}$ be differentiable. If $f'(x) \neq 0$ for all $x \in [a,b]$, show that either $f'(x) \geq 0 \ \forall x \in [a,b]$ or $f'(x) \leq 0 \ \forall x \in [a,b]$.

11) Let $f:[a,b] \to \mathbb{R}$ be differentiable and let $\alpha \in \mathbb{R}$ be such that $f'(a) < \alpha < f'(b)$. Define $g(x) = f(x) - \alpha x$ for all $x \in [a,b]$.

a) Show that there exists $c \in [a,b]$ such that $g'(c) = 0$.
Hint: prove by contradiction, noting that $g'(a) < 0$ and $g'(b) < 0$.

b) From the above, conclude that if a function f is differentiable on an interval $[a,b]$, then f' has the Intermediate Value Property on $[a,b]$.

12) Let f be differentiable on $[a,b]$ $(a<b)$. Show that there exist $c_1, c_2, c_3 \in (a,b)$ such that $c_2 \neq c_3$ and $f'(c_2) + f'(c_3) = 2f'(c_1)$.

13) Suppose $f:[0,1] \to \mathbb{R}$ is continuous and $\int_0^1 f(t)dt = 1$.

a) Show that there exists $c \in (0,1)$ such that $f(c) = 1$.

b) Show that there exist $c_1 \neq c_2$ in $(0,1)$ such that $f(c_1) + f(c_2) = 2$.

14) Let $f:[0,1] \to \mathbb{R}$ be such that $|f'(x)| < 10$ for all $x \in (0,1)$ and let (x_n) be a sequence in $(0,1)$ satisfying the Cauchy criterion. Show that the sequence $\left(f(x_n)\right)$ converges.

15) Let $f:[0,1]\to[0,1]$ be such that $f'(x)<0$ for all $x\in[0,1]$. Show that there is one and only one $c\in[0,1]$ such that $f(c)=c^2$.

16) Let $f:[0,1]\to\mathbb{R}$ and $a_n=f\left(\frac{1}{n}\right)-f\left(\frac{1}{n+1}\right),\ n=1,2,\ldots$

Show that:

a) if f is continuous, then $\sum_{n=1}^{\infty}a_n$ converges;

b) if f is differentiable and $|f'(x)|<\frac{1}{2}\ \forall x\in[0,1]$, then $\sum_{n=1}^{\infty}a_n(\cos n)\sqrt{n}$ converges.

Power Series, Taylor Series (טורי חזקות וטורי טיילור)

Questions

1) Answer the following sections:

a) Let $f:[a,b]\to\mathbb{R}$ be such that $f''(x)\geq 0$ for all $x\in[a,b]$.

Suppose $x_0\in[a,b]$.

Show that for any $x\in[a,b]$ $f(x)\geq f(x_0)+f'(x_0)(x-x_0)$.

I.e., the graph of f lies above the tangent line to the graph at $(x_0,f(x_0))$.

b) Show that $\cos y-\cos x\geq(x-y)\sin x$ for all $x,y\in[\frac{\pi}{2},\frac{3\pi}{2}]$.

2) Let $f:(a,b)\to\mathbb{R}$ be infinitely differentiable and let $x_0\in(a,b)$. Suppose that there exists $M>0$ such that $\left|f^{(n)}(x)\right|\leq M^n$ for all $n\in\mathbb{N}$ and $x\in(a,b)$.

Show that Taylor's series of f around x_0 converges to $f(x)$ for all $x\in(a,b)$.

3) Let (a_n) be a sequence of nonnegative reals and suppose that $(a_n^{\frac{1}{n}})$ is a bounded sequence. For each n, define $A_n=\sup\{a_k^{\frac{1}{k}}:k\geq n\}$.

(A_n) converges since it is decreasing and bounded below (by 0).

So $A_n\to L$ for some $L\geq 0$.

a) Show that if $L<1$, the series $\sum_{n=1}^{\infty}a_n$ converges and if $L>1$ the series diverges.

b) Show that the radius of convergence of the power series $\sum_{n=1}^{\infty}a_nx^n$ is $\dfrac{1}{L}$.