

חדוא 1

פרק 39 - תרגילי תיאוריה מתקדמים - חשבון דיפרנציאלי (הפרק באנגלית)

תוכן העניינים

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Convergence of a Sequence, Monotone Sequences (סדרות)

Questions

- 1) Let A be a non-empty subset of $\mathbb{R}$ and $\alpha = \inf A$. Show that there exists a sequence (a_n) such that an $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \rightarrow \alpha$.
- 2) Let $x_0 \in \mathbb{Q}$. Show that there exists a sequence (x_n) of irrational numbers such that $x_n \rightarrow x_0$.
- 3) Let A be a non-empty subset of $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Show that there exists a sequence (a_n) in A such that $|x_0 - a_n| \rightarrow d(x_0, A)$. Recall that $d(x, A) = \inf \{|x - a| : a \in A\}$.
- 4) Let (a_k) be a bounded sequence. For every $n \in \mathbb{N}$, define $x_n = \sup\{a_k : k < n\}$. Show that the sequence (x_n) converges.

Cauchy Criterion, Bolzano - Weierstrass Theorem

- 5) Let (x_n) be a sequence of integers such that $|x_{n+1} - x_n| \geq 1$ for all $n \in \mathbb{N}$.
Prove or disprove the following statements.
 - a) The sequence (x_n) does not satisfy the Cauchy criterion.
 - b) The sequence (x_n) cannot have a convergent subsequence.
- 6) Show that a sequence (x_n) of real numbers has no convergent subsequence if and only if $|x_n| \rightarrow \infty$.
- 7) Let (x_n) be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose that every subsequence of (x_n) has a subsequence converging to x_0 . Show that $x_n \rightarrow x_0$.

- 8) Let (x_n) be a sequence in $\mathbb{R}$. We say that a positive integer n is a peak of the sequence if $m > n$ implies $x_n > x_m$ (i.e., if x_n is greater than every subsequent term in the sequence).
- a) If (x_n) has infinitely many peaks, show that it has a decreasing subsequence.
 - b) If (x_n) has only finitely many peaks, show that it has an increasing subsequence.
 - c) From (a) and (b) conclude that every sequence in $\mathbb{R}$ has a monotone subsequence. Further, every bounded sequence in $\mathbb{R}$ has a convergent subsequence (An alternate proof of Bolzano-Weierstrass Theorem).

Continuity and Limits (גבולות ורציפות)

Questions

- 1) Let $\lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 5$. Show that $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 0$.
- 2) Let $f: \mathbb{R} \rightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose $\lim_{x \rightarrow x_0} f(x)$ exists.
Show that $\lim_{x \rightarrow 0} f(x + x_0) = \lim_{x \rightarrow x_0} f(x)$.
- 3) Let $f(x) = |x|$ for every $x \in \mathbb{R}$. Show that f is continuous on $\mathbb{R}$.
- 4) Let $f: [0, \pi] \rightarrow \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = x \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$ for $x \neq 0$.
Is f continuous?
- 5) Let $[\cdot]$ denote the integer part function and $f: [0, \infty) \rightarrow \mathbb{R}$ be defined by $f(x) = [x^2] \sin \pi x$.
 - a) Show that f is continuous at each $x \neq \sqrt{n}$, $n \in \mathbb{N}$. [Here $\mathbb{N}$ includes 0]
 - b) Show that f is continuous at each $x = k \in \mathbb{N}$.
 - c) Show that f is discontinuous at each $x = \sqrt{n}$, $n \in \mathbb{N}$ such that $x \notin \mathbb{N}$.
- 6) Let the function $f: [0, 1] \rightarrow [a, b]$ be one-one and onto. Suppose f is continuous.
Show that f^{-1} is also continuous.
- 7) Let $f: (0, 1) \rightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q \in \mathbb{N} \text{ and } p, q \text{ have no common factor} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$
 - a) Suppose $x_n \rightarrow x_0$ for some x_0 , with $x_n \neq x$ for all $n \in \mathbb{N}$, and suppose $x_n = \frac{p_n}{q_n} \in (0, 1)$ where $p_n, q_n \in \mathbb{N}$ have no common factors. Show that $\lim_{n \rightarrow \infty} q_n = \infty$.
 - b) Show that f is continuous at every irrational.
 - c) Show that f is discontinuous at every rational.

Existence of Extrema, Intermediate Value Property (משפט ערך הביניים ומשפט ויירשטראס)

Questions

- 1) Give an example of a function f on $[0,1]$ which is not continuous but satisfies the IVP*. *We say that f has the property IVP [Intermediate Value Property] on $[a,b]$ if for every $x, y \in [a,b]$ and α satisfying $f(x) < \alpha < f(y)$ or $f(x) > \alpha > f(y)$ there exists $x_0 \in [x, y]$, such that $f(x_0) = \alpha$.
- 2) Let $f : [0,1] \rightarrow \mathbb{R}$ be continuous. Show that there exists an $x_0 \in [0,1]$ such that $f(x_0) = \frac{1}{3} \left[f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) \right]$.
- 3) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Show that f is a constant function if
 - a) $f(x)$ is rational for each $x \in \mathbb{R}$.
 - b) $f(x)$ is an integer for each $x \in \mathbb{Q}$.
- 4) Let $p(x) : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial function of odd degree. Show that p is onto.
- 5) Let $f, g : [0,1] \rightarrow \mathbb{R}$ be continuous such that $\inf\{f(x) : x \in [0,1]\} = \inf\{g(x) : x \in [0,1]\}$.
Show that there exists $x_0 \in [0,1]$ such that $f(x_0) = g(x_0)$.
- 6) A cross country runner runs continuously an eight kilometers course in 40 minutes without taking rest. Show that, somewhere along the course, the runner must have covered a distance of one kilometer in exactly 5 minutes.
- 7) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous one-one map.
Show that f is either strictly increasing or strictly decreasing.
- 8) Let $f : \mathbb{R} \rightarrow [0, \infty)$ be a bijective map. Show that f is not continuous on $\mathbb{R}$.
- 9) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function.
 - a) Suppose f attains each of values exactly two times. Given:
 $f(x_1) = f(x_2) = \alpha$ for some $x_1, x_2, \alpha \in \mathbb{R}$, and $f(x_0) > \alpha$ for some $x_0 \in [x_1, x_2]$. Show that f attains its maximum in $[x_1, x_2]$ exactly at one point.
 - b) Using (a) show that f cannot attain each of its values exactly two times.

Differentiability and Rolle's Theorem

(גזירות ומשפט רול)

Questions

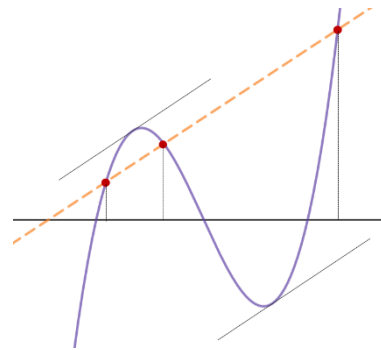
- 1) Let $f : (-1, 2) \rightarrow \mathbb{R}$ be a twice differentiable function such that $f''(0) > 0$.
Show that there exists $n \in \mathbb{N}$ that $f(\frac{1}{n}) \neq 1$.
- 2) Let f be a differentiable function on $\mathbb{R}$ such that $f(0) = f(1) = 0$, $f'(0) > 0$ and $f'(1) > 0$.
 - a) Show that $\exists \delta > 0$ such that $f(x) < 0$ on $(1-\delta, 1)$.
 - b) Show that $\exists c_1, c_2 \in (0, 1)$ such that $c_1 \neq c_2$ and $f'(c_1) = f'(c_2) = 0$.
- 3) Let $f : (0, 1) \rightarrow \mathbb{R}$ be thrice differentiable. Suppose $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. Show that there exists $x_0 \in (0, 1)$ such that $f'''(x_0) = 0$.
- 4) Give an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is differentiable only at $x = 1$.
- 5) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$.
 - a) If $f(x_0) \neq 0$, show that $|f|$ is also differentiable at x_0 .
 - b) If $f(x_0) = 0$, give examples to show that $|f|$ may or may not be differentiable at x_0 .
- 6) Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$. Define $h(x) = \max\{f(x), g(x)\} \forall x \in \mathbb{R}$. Show that if $f(x_0) \neq g(x_0)$ then h is differentiable at x_0 .
- 7) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable at $x = 1$ and $f(1) = 1$. Show that if $k \in \mathbb{N}$ then
$$\lim_{n \rightarrow \infty} n \left[f\left(1 + \frac{1}{n}\right) + f\left(1 + \frac{2}{n}\right) + \dots + f\left(1 + \frac{k}{n}\right) - k \right] = \frac{k(k+1)}{2} f'(1).$$
- 8) Let $f : [0, 1] \rightarrow \mathbb{R}$ be differentiable and $f(0) = 0$ and $f(1) = 1$.
Show that the equation $f'(x) = 2x$ has a solution on $(0, 1)$.

- 9) Let $f : [a, b] \rightarrow \mathbb{R}$ be such that $f'''(x)$ exists for all $x \in [a, b]$.
 Suppose that $f(a) = f(b) = f'(a) = f'(b) = 0$.
 Show that the equation $f'''(x) = 0$ has a solution.
- 10) Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable functions. Suppose that
 $\forall x \in \mathbb{R}, f'(x)g(x) \neq f(x)g'(x)$. Show that between any two roots of f , there
 exists at least one root* of g .
 *Recall: a root of f is a solution of $f(x) = 0$.
- 11) Let $f : (0, \infty) \rightarrow \mathbb{R}$ satisfy $f(xy) = f(x) + f(y) \quad \forall x, y \in (0, \infty)$.
 Suppose that f is differentiable at $x = 1$.
 Show that f is differentiable at every $x \in (0, \infty)$ and $f'(x) = \frac{f'(1)}{x}$.
 Hint: first show that $f(1) = 0$ and $f(\frac{x}{y}) = f(x) - f(y)$.
- 12) Let $p(x) = a + bx + cx^2$. Find all values of $a, b, c \in \mathbb{R}$ for which the function
 $p(|x|)$ is differentiable at 0.

Mean Value Theorem, L'Hôpital's Rule (משפט לגראנז' וכלל לופיטל)

Questions

- Does there exist a differentiable function $f:[0,2] \rightarrow \mathbb{R}$ satisfying $f(0) = -1$, $f(2) = 4$ and $f'(x) \leq 2$, for all $x \in [0,2]$?
- Let $f:[0,1] \rightarrow \mathbb{R}$ be differentiable such that $|f'(x)| < 1$ for all $x \in [0,1]$. Show that there exists at most one $c \in [0,1]$ such that $f(c) = c$.
- Let $f:\mathbb{R} \rightarrow \mathbb{R}$ be differentiable such that for some $\alpha \in \mathbb{R}$, $|f'(x)| \leq \alpha < 1$ for all $x \in \mathbb{R}$. Let $a_1 \in \mathbb{R}$ and define a sequence (a_n) recursively by $a_{n+1} = f(a_n)$. Show that (a_n) converges.
- Let $f:[0,1] \rightarrow \mathbb{R}$ be twice differentiable. Suppose that the line segment joining the points $(0, f(0))$ and $(1, f(1))$ intersect the graph of f at a point $(a, f(a))$, where $0 < a < 1$. Show that there exists $x_0 \in [0,1]$ such that $f''(x_0) = 0$.
- Let $f:[0,1] \rightarrow \mathbb{R}$ be continuous. Suppose f is differentiable on $(0,1)$ and $\lim_{x \rightarrow 0^+} f'(x) = L$ for some $L \in \mathbb{R}$. Show that $f'(0)$ exists and $f'(0) = L$.
- Let $f:[0,1] \rightarrow \mathbb{R}$ be differentiable and $f(0) = 0$. Suppose that $|f'(x)| < |f(x)|$ for all $x \in [0,1]$. Show that $f(x) = 0$ for all $x \in [0,1]$.
- Let $f:[0,\infty) \rightarrow \mathbb{R}$ be continuous and $f(0) = 0$. Suppose that $f'(x)$ exists for all $x \in (0,\infty)$ and f' is increasing on $(0,\infty)$. Show that the function $g(x) = \frac{f(x)}{x}$ is increasing on $(0,\infty)$.



- 8) Let $a \geq 0$ and $f : [a, b] \rightarrow \mathbb{R}$ be differentiable. Using Cauchy's mean value theorem*, show that there exist $c_1, c_2 \in (a, b)$ such that $\frac{f'(c_1)}{a+b} = \frac{f'(c_2)}{2c_2}$.
- * $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \quad a < c < b$
- 9) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that $f''(c)$ exists at some $c \in \mathbb{R}$. Using L'Hôpital's rule, show that $\lim_{h \rightarrow 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c)$. Give an example where the above limit exists but $f''(c)$ does not exist.
- 10) Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable. If $f'(x) \neq 0$ for all $x \in [a, b]$, show that either $f'(x) \geq 0 \quad \forall x \in [a, b]$ or $f'(x) \leq 0 \quad \forall x \in [a, b]$.
- 11) Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable and let $\alpha \in \mathbb{R}$ be such that $f'(a) < \alpha < f'(b)$. Define $g(x) = f(x) - \alpha x$ for all $x \in [a, b]$.
- Show that there exists $c \in [a, b]$ such that $g'(c) = 0$.
Hint: prove by contradiction, noting that $g'(a) < 0$ and $g'(b) < 0$.
 - From the above, conclude that if a function f is differentiable on an interval $[a, b]$, then f' has the Intermediate Value Property on $[a, b]$.
- 12) Let f be differentiable on $[a, b]$ ($a < b$). Show that there exist $c_1, c_2, c_3 \in (a, b)$ such that $c_2 \neq c_3$ and $f'(c_2) + f'(c_3) = 2f'(c_1)$.
- 13) Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is continuous and $\int_0^1 f(t) dt = 1$.
- Show that there exists $c \in (0, 1)$ such that $f(c) = 1$.
 - Show that there exist $c_1 \neq c_2$ in $(0, 1)$ such that $f(c_1) + f(c_2) = 2$.
- 14) Let $f : [0, 1] \rightarrow \mathbb{R}$ be such that $|f'(x)| < 10$ for all $x \in (0, 1)$ and let (x_n) be a sequence in $(0, 1)$ satisfying the Cauchy criterion. Show that the sequence $(f(x_n))$ converges.

15) Let $f : [0,1] \rightarrow [0,1]$ be such that $f'(x) < 0$ for all $x \in [0,1]$. Show that there is one and only one $c \in [0,1]$ such that $f(c) = c^2$.

16) Let $f : [0,1] \rightarrow \mathbb{R}$ and $a_n = f\left(\frac{1}{n}\right) - f\left(\frac{1}{n+1}\right)$, $n=1,2,\dots$

Show that:

a) if f is continuous, then $\sum_{n=1}^{\infty} a_n$ converges;

b) if f is differentiable and $|f'(x)| < \frac{1}{2} \forall x \in [0,1]$, then

$\sum_{n=1}^{\infty} a_n (\cos n) \sqrt{n}$ converges.

Power Series, Taylor Series (טורי חזקות וטורי טיילור)

Questions

1) Answer the following sections:

a) Let $f:[a,b] \rightarrow \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a,b]$.

Suppose $x_0 \in [a,b]$.

Show that for any $x \in [a,b]$ $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$.

I.e., the graph of f lies above the tangent line to the graph at $(x_0, f(x_0))$.

b) Show that $\cos y - \cos x \geq (x - y)\sin x$ for all $x, y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

2) Let $f:(a,b) \rightarrow \mathbb{R}$ be infinitely differentiable and let $x_0 \in (a,b)$. Suppose that there exists $M > 0$ such that $|f^{(n)}(x)| \leq M^n$ for all $n \in \mathbb{N}$ and $x \in (a,b)$.

Show that Taylor's series of f around x_0 converges to $f(x)$ for all $x \in (a,b)$.

3) Let (a_n) be a sequence of nonnegative reals and suppose that $(a_n^{\frac{1}{n}})$ is a

bounded sequence. For each n , define $A_n = \sup\{a_k^{\frac{1}{k}} : k \geq n\}$.

(A_n) converges since it is decreasing and bounded below (by 0).

So $A_n \rightarrow L$ for some $L \geq 0$.

a) Show that if $L < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges and if $L > 1$ the series diverges.

b) Show that the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$ is $\frac{1}{L}$.